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Comparison of Various Formulations of Three-Phase Flow in Porous Media[1]

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Various formulations of the governing equations that describe three-phase (e.g., water, oil, and gas) flow in porous media, including phase, global, and pseudo-global pressure-saturation formulations, are discussed in this paper. Comparisons of these differential formulations are theoretically and numerically presented for the first time. It is shown that the global pressure-saturation formulation is the most efficient one from the computational point of view in the case where the three-phase relative permeability and capillary pressure curves satisfy a so-called total differential condition, the pseudo-global formulation is useful when the fractional flow functions of the water and gas phases are close to their respective mean values, and the phase formulation can be applied generally. © 1997 Academic Press

1. INTRODUCTION

It has been shown that the governing equations describing two-phase flow in porous media can be written in a fractional flow formulation, i.e., in terms of a global pressure and saturation [1, 8, 14]. Further, it has been proven that this fractional flow approach is far more efficient than the original two-pressure approach from the computational point of view [5, 11, 13]. The main reasons for this are that the differential equations written in the fractional flow formulation formally resemble the governing equations for single-phase flow, and that efficient numerical schemes can be devised to take advantage of many physical properties inherent in the flow equations.

In this paper we discuss various formulations of the governing equations describing three-phase (e.g., water, oil, and gas) flow in porous media. We show that, under a so-called total differential condition on the shape of three-phase relative permeability and capillary pressure functions, the governing equations can be written in a fractional flow formulation, i.e., in terms of a global pressure and two saturations. The case of three-phase flow is quite different from the case of two-phase flow. The three-phase relative permeability and capillary pressure curves are far more complex than the corresponding two-phase curves. It is the complexity of these three-phase curves that complicates the derivation of the global pressure-saturation form for the former case. In the two-phase flow, the governing equations can be written in terms of a global pressure and saturation without any hypothesis [1, 8, 14]. However, in the three-phase flow we show that the total differential condition is necessary and sufficient for the governing equations to be written in terms of a global pressure and two saturations. While this condition is not satisfied for all the existing three-phase curves, it is here verified that it is satisfied for some simplified models.

For the above reason on the total differential condition, we also derive other formulations of the governing equations for three-phase flow in porous media. We show that these equations can be written in terms of a phase or pseudo-global pressure and two saturations without any assumption. However, it turns out that the phase and pseudo-global pressure-saturation forms are much more complicated than the global pressure-saturation form. In particular, the coupling between the pressure and saturation equations in the phase and pseudo-global pressure-saturation forms is stronger, and thus these equations are more expensive to solve. This agrees with our theoretical and numerical observations, which are carried out here for the first time. The pseudo-global formulation is useful when the total differential condition is violated and the fractional flow functions of the water and gas phases are close to their respective mean values. In this case the pressure equation is more decoupled from the saturation equations in this formulation than in the phase formulation. In the general case where these two features are not satisfied, the phase formulation can be applied.

In the next section we review the governing equations for three-phase flow in a porous medium. Then the phase, global, and pseudo-global pressure-saturation forms with

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a total velocity and flux are derived in Sections 3–5, respectively. A theoretical comparison of these forms is presented in Section 6. A comparison between the global and phase forms, and between the phase and pseudo-global forms via numerical experiments is given in Section 7; finite element and difference methods are applied to solve the partial differential equations. Finally, a concluding remark is given in Section 8.

2. THE GOVERNING EQUATIONS

The usual equations describing the flow of three immiscible fluids in a porous medium $\Omega \subset \Re^3$ are given by the mass balance equation and Darcy's law for each of the fluid phases [4, 21],

$$\frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t} + \nabla\cdot(\rho_\alpha u_\alpha) = q_\alpha, \qquad x\in\Omega, t>0, \quad (2.1a)$$

$$u_\alpha = -\frac{kk_{r\alpha}}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha\tilde{g}), \qquad x\in\Omega, t>0, \quad (2.1b)$$

where ϕ and k are the porosity and absolute permeability of the porous medium; ρ_α, s_α, p_α, u_α, and μ_α are, respectively, the density, (reduced) saturation, pressure, volumetric velocity, and viscosity of the α-phase; q_α is the source/sink term; $k_{r\alpha}$ is the relative permeability of the α-phase; and $\tilde{g}$ is the gravitational, downward-pointing, constant vector. Below $\alpha = w$, o, and g denote water, oil, and gas phases, respectively, for example. In addition to (2.1), we also have the customary property for the saturations,

$$\sum_\alpha s_\alpha = 1, \quad (2.2)$$

where (and later) $\Sigma_\alpha = \Sigma_{\alpha=w,o,g}$, and define, for notational convenience, the capillary pressure functions,

$$p_{c\alpha o} = p_\alpha - p_o, \qquad \alpha = w, o, g, \quad (2.3)$$

where $p_{coo} \equiv 0$, p_{cgo} represents the gas phase capillary pressure, and p_{cwo} is the negative water phase capillary pressure.

The dependent variables are s_α, p_α, and u_α. In (2.1) and (2.2), we have utilized the reduced saturations s_α, which are related to the phase saturations $\tilde{s}_\alpha$ by

$$s_\alpha = \frac{\tilde{s}_\alpha - \tilde{s}_{r\alpha}}{1 - \tilde{s}_{rw} - \tilde{s}_{ro} - \tilde{s}_{rg}}, \qquad \alpha = w, o, g, \quad (2.4)$$

where $\tilde{s}_{r\alpha}$ is the residual saturation of the α-phase, $\alpha = w$, o, g. The porosity ϕ can be a function of space and pressures, and the absolute permeability k can depend on space and any dependent variables. The density ρ_α and viscosity μ_α are functions of pressures. Finally, we assume that the capillary pressure and relative permeability functions depend upon the saturations s_α solely. For notational simplicity, we neglect their dependence on space, which would then introduce lower-order terms in the partial differential equations [8, 14]. In the next three sections we shall write Eqs. (2.1)–(2.3) in terms of a pressure p and the two saturations s_w and s_g.

3. PHASE FORMULATION

In this section the phase pressure-saturation formulation with a total velocity and flux is derived.

3.1. *Phase Formulation with a Total Velocity.*

For expositional convenience, we introduce the phase mobility functions

$$\lambda_\alpha = k_{r\alpha}/\mu_\alpha, \qquad \alpha = w, o, g,$$

and the total mobility

$$\lambda = \sum_\alpha \lambda_\alpha.$$

Also, we define the fractional flow functions

$$f_\alpha = \lambda_\alpha/\lambda, \qquad \alpha = w, o, g.$$

We see that $\Sigma_\alpha f_\alpha = 1$.

We use the oil phase pressure as the pressure variable in this section,

$$p = p_o, \quad (3.1)$$

and define the total velocity

$$u = \sum_\alpha u_\alpha. \quad (3.2)$$

Then, use (3.1) and (3.2), carry out the differentiation indicated in (2.1a), divide by ρ_α in (2.1a), and apply (2.2) and (2.3) to obtain the differential equations with $x \in \Omega$ and $t > 0$,

$$u = -k\lambda(\nabla p - G_\lambda + \sum_\alpha f_\alpha \nabla p_{c\alpha o}), \quad (3.3a)$$

$$\nabla\cdot u = -\frac{\partial\phi}{\partial t} + \sum_\alpha \frac{1}{\rho_\alpha}\left(q_\alpha - \phi s_\alpha\frac{\partial\rho_\alpha}{\partial t} - u_\alpha\cdot\nabla\rho_\alpha\right), \quad (3.3b)$$

and

$$\phi\frac{\partial s_\alpha}{\partial t}+\nabla\cdot\left\{f_\alpha u+kf_\alpha\sum_\beta\lambda_\beta(\nabla(p_{c\beta o}-p_{c\alpha o})-(\rho_\beta-\rho_\alpha)\tilde{g})\right\}$$
$$=-s_\alpha\frac{\partial\phi}{\partial t}+\frac{1}{\rho_\alpha}\left(q_\alpha-\phi s_\alpha\frac{\partial\rho_\alpha}{\partial t}-u_\alpha\cdot\nabla\rho_\alpha\right),\qquad\alpha=w,g,\tag{3.4}$$

where

$$G_\lambda=\tilde{g}\sum_\alpha f_\alpha\rho_\alpha.$$

The equations in (3.3) and (3.4) are, respectively, the pressure and saturation equations. The phase velocity is related to the total velocity by

$$\begin{aligned}u_\alpha=f_\alpha u+kf_\alpha\sum_\beta\lambda_\beta(\nabla(p_{c\beta o}-p_{c\alpha o})\\-(\rho_\beta-\rho_\alpha)\tilde{g}),\qquad\alpha=w,o,g.\end{aligned}\tag{3.5}$$

3.2. *Phase Formulation with a Total Flux.* In the right-hand sides of (3.3b) and (3.4) appear the terms $u_\alpha\cdot\nabla\rho_\alpha$, which are essentially quadratic in the velocities. To get rid of these terms, we now introduce a total flux. Toward that end, set

$$\lambda_\alpha=k_{r\alpha}\rho_\alpha/\mu_\alpha,\qquad\lambda=\sum_\beta\lambda_\beta,\quad\alpha=w,o,g,$$

and

$$f_\alpha=\lambda_\alpha/\lambda,\qquad\alpha=w,o,g.$$

The pressure variable is defined as in (3.1), but a total flux is now introduced:

$$u=\sum_\alpha\rho_\alpha u_\alpha.\tag{3.6}$$

Then with the same manipulation on (2.1) as above, we have the pressure and saturation equations with $x\in\Omega$ and $t>0$:

$$u=-k\lambda(\nabla p-G_\lambda+\sum_\alpha f_\alpha\nabla p_{c\alpha o}),\tag{3.7a}$$

$$\sum_\alpha\frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t}+\nabla\cdot u=\sum_\alpha q_\alpha,\tag{3.7b}$$

and

$$\frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t}+\nabla\cdot\left\{f_\alpha u+kf_\alpha\sum_\beta\lambda_\beta(\nabla(p_{c\beta o}-p_{c\alpha o})\right.$$
$$\left.-(\rho_\beta-\rho_\alpha)\tilde{g})\right\}=q_\alpha,\qquad\alpha=w,g.$$

The phase velocity is given by

$$u_\alpha=\rho_\alpha^{-1}\left\{f_\alpha u+kf_\alpha\sum_\beta\lambda_\beta(\nabla(p_{c\beta o}-p_{coo})\right.$$
$$\left.-(\rho_\beta-\rho_\alpha)\tilde{g})\right\}\qquad\alpha=w,o,g.$$

4. GLOBAL FORMULATION

In this section the global pressure-saturation formulation with the total velocity and flux is obtained.

4.1. *Global Formulation with the Total Velocity.* The phase and total mobilities and the fractional flow functions are defined in the same manner as in Subsection 3.1; i.e.,

$$\lambda_\alpha=k_{r\alpha}/\mu_\alpha,\qquad\lambda=\sum_\beta\lambda_\beta,\quad f_\alpha=\lambda_\alpha/\lambda,\quad\alpha=w,o,g.$$

To introduce a global pressure, we assume that the fractional flow functions f_α depend solely on the saturations s_w and s_g (for pressure-dependent functions f_α, see the next subsection), and that there exists a function $(s_w,s_g)\mapsto p_c(s_w,s_g)$ such that

$$\nabla p_c=f_w\nabla p_{cwo}+f_g\nabla p_{cgo}.\tag{4.1}$$

This holds if and only if the following equations are satisfied:

$$\frac{\partial p_c}{\partial s_w}=f_w\frac{\partial p_{cwo}}{\partial s_w}+f_g\frac{\partial p_{cgo}}{\partial s_w},\tag{4.2a}$$

$$\frac{\partial p_c}{\partial s_g}=f_w\frac{\partial p_{cwo}}{\partial s_g}+f_g\frac{\partial p_{cgo}}{\partial s_g}.\tag{4.2b}$$

A necessary and sufficient condition for existence of a function p_c satisfying (4.2) is

$$\frac{\partial f_w}{\partial s_g}\frac{\partial p_{cwo}}{\partial s_w}+\frac{\partial f_g}{\partial s_g}\frac{\partial p_{cgo}}{\partial s_w}=\frac{\partial f_w}{\partial s_w}\frac{\partial p_{cwo}}{\partial s_g}+\frac{\partial f_g}{\partial s_w}\frac{\partial p_{cgo}}{\partial s_g}.\tag{4.3}$$

This condition is referred to as the total differential condition [8]. When the condition (4.3) is satisfied, the function p_c is determined by

$$\begin{aligned} p_c(s_w, s_g) = \int_1^{s_w} \left\{ f_w(\xi,0) \frac{\partial p_{cwo}}{\partial s_w}(\xi,0) + f_g(\xi,0) \frac{\partial p_{cgo}}{\partial s_w}(\xi,0) \right\} d\xi \\ + \int_0^{s_g} \left\{ f_w(s_w,\xi) \frac{\partial p_{cwo}}{\partial s_g}(s_w,\xi) + f_g(s_w,\xi) \frac{\partial p_{cgo}}{\partial s_g}(s_w,\xi) \right\} d\xi, \end{aligned} \tag{4.4}$$

where we assume that the above integrals are well-defined, which is always true in practical situations [8]. We now introduce the global pressure by

$$p = p_o + p_c, \tag{4.5}$$

and the total velocity by

$$u = \sum_\alpha u_\alpha. \tag{4.6}$$

Now, use the condition (4.3), the definitions in (4.4)–(4.6), and the same calculations as in Subsection 3.1 to get the pressure and saturation equations with $x \in \Omega$ and $t > 0$,

$$u = -k\lambda(\nabla p - G_\lambda), \tag{4.7a}$$

$$\nabla \cdot u = -\frac{\partial \phi}{\partial t} + \sum_\alpha \frac{1}{\rho_\alpha} \left(q_\alpha - \phi s_\alpha \frac{\partial \rho_\alpha}{\partial t} - u_\alpha \cdot \nabla \rho_\alpha \right), \tag{4.7b}$$

and

$$\begin{aligned} \phi \frac{\partial s_\alpha}{\partial t} + \nabla \cdot \{ f_\alpha u + k\lambda_\alpha (\nabla(p_c - p_{c\alpha o}) - \delta_\alpha) \} = -s_\alpha \frac{\partial \phi}{\partial t} \\ + \frac{1}{\rho_\alpha} \left(q_\alpha - \phi s_\alpha \frac{\partial \rho_\alpha}{\partial t} - u_\alpha \cdot \nabla \rho_\alpha \right), \qquad \alpha = w, g, \end{aligned} \tag{4.8}$$

where

$$\delta_\alpha = (f_\beta(\rho_\beta - \rho_\alpha) + f_\gamma(\rho_\gamma - \rho_\alpha))\tilde{g},$$
$$\alpha, \beta, \gamma = w, o, g, \alpha \neq \beta, \beta \neq \gamma, \gamma \neq \alpha.$$

Finally, the phase velocity is determined by

$$u_\alpha = f_\alpha u + k\lambda_\alpha(\nabla(p_c - p_{c\alpha o}) - \delta_\alpha), \qquad \alpha = w, o, g. \tag{4.9}$$

4.2. *Global Formulation with the Total Flux.* As in Subsection 3.2, to get rid of the quadratic terms in the velocities in (4.7b), we define

$$\lambda_\alpha = k_{r\alpha}\rho_\alpha/\mu_\alpha, \qquad \lambda = \sum_\beta \lambda_\beta, \quad \alpha = w, o, g,$$

and

$$f_\alpha = \lambda_\alpha/\lambda, \qquad \alpha = w, o, g.$$

Also, define the total flux

$$u = \sum_\alpha \rho_\alpha u_\alpha. \tag{4.10}$$

In the present case we assume that the fractional flow functions f_α depend on the saturations s_w and s_g and a pressure p, and that there exists a function $(s_w, s_g, p) \mapsto p_c(s_w, s_g, p)$ satisfying

$$\nabla p_c = f_w \nabla p_{cwo} + f_g \nabla p_{cgo} + \frac{\partial p_c}{\partial p} \nabla p. \tag{4.11}$$

The assumption on the dependence on the pressure p means that we ignore the error caused by calculating the density and viscosity functions for the α-phase at p instead of p_α. For details on this error, the reader is referred to [14] for a similar treatment in the two-phase flow.

With the same argument as in Subsection 4.1, a necessary and sufficient condition for existence of a function p_c satisfying (4.11) is (4.3); i.e.,

$$\frac{\partial f_w}{\partial s_g} \frac{\partial p_{cwo}}{\partial s_w} + \frac{\partial f_g}{\partial s_g} \frac{\partial p_{cgo}}{\partial s_w} = \frac{\partial f_w}{\partial s_w} \frac{\partial p_{cwo}}{\partial s_g} + \frac{\partial f_g}{\partial s_w} \frac{\partial p_{cgo}}{\partial s_g}, \tag{4.12}$$

where p is treated as a parameter. Under the condition (4.12), the function p_c is described by

$$\begin{aligned} p_c(s_w, s_g, p) = \int_1^{s_w} \Big\{ f_w(\xi, 0, p) \frac{\partial p_{cwo}}{\partial s_w}(\xi, 0) \\ + f_g(\xi, 0, p) \frac{\partial p_{cgo}}{\partial s_w}(\xi, 0) \Big\} d\xi \\ + \int_0^{s_g} \Big\{ f_w(s_w, \xi, p) \frac{\partial p_{cwo}}{\partial s_g}(s_w, \xi) \\ + f_g(s_w, \xi, p) \frac{\partial p_{cgo}}{\partial s_g}(s_w, \xi) \Big\} d\xi. \end{aligned} \tag{4.13}$$

The global pressure is again defined by

$$p = p_o + p_c. \tag{4.14}$$

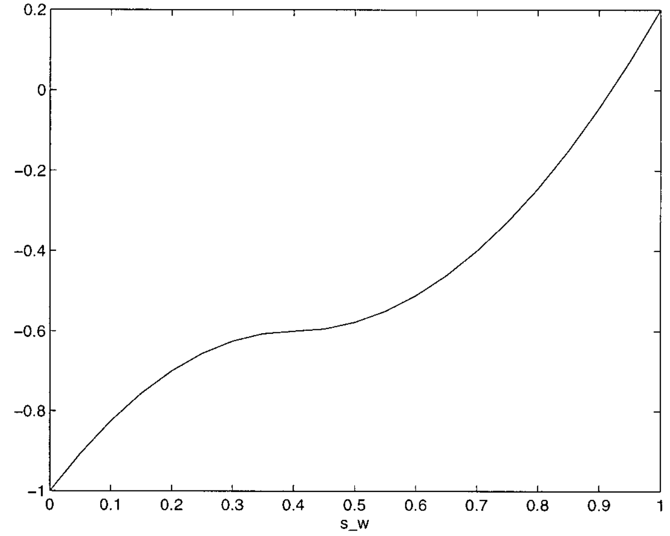

FIG. 1. The typical normalized capillary pressure p_{cwo}.

Then, as before, we have the pressure and saturation equations with $x \in \Omega$ and $t > 0$,

$$u = -k\lambda(\omega\nabla p - G_\lambda), \tag{4.15a}$$

$$\sum_\alpha \frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot u = \sum_\alpha q_\alpha, \tag{4.15b}$$

and

$$\frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot \Big\{ \omega^{-1} f_\alpha u + k\lambda_\alpha(\nabla(p_c - p_{c\alpha o}) - \delta_\alpha) - \omega^{-1}\frac{\partial p_c}{\partial p} G_\lambda \Big\} = q_\alpha, \qquad \alpha = w, g, \tag{4.16}$$

where

$$\omega(s_w, s_g, p) = 1 - \frac{\partial p_c}{\partial p}.$$

The phase velocity is computed by

$$u_\alpha = \rho_\alpha^{-1} \Big\{ \omega^{-1} f_\alpha u + k\lambda_\alpha(\nabla(p_c - p_{c\alpha o}) - \delta_\alpha) - \omega^{-1}\frac{\partial p_c}{\partial p} G_\lambda \Big\}, \qquad \alpha = w, o, g. \tag{4.17}$$

4.3. *Total Differential Condition.* In this subsection we discuss the total differential condition (4.3). For three-phase flow, the classical capillary pressures [19] are normally used:

$$p_{cwo} = p_{cwo}(s_w), \qquad p_{cgo} = p_{cgo}(s_g). \tag{4.18}$$

Typical normalized capillary pressure functions are shown in Figs. 1 and 2.

Using (4.18), the condition (4.3) reduces to

$$\frac{\partial f_w}{\partial s_g}\,\frac{\partial p_{cwo}}{\partial s_w} = \frac{\partial f_g}{\partial s_w}\,\frac{\partial p_{cgo}}{\partial s_g}. \tag{4.19}$$

Also, we have the usual definitions of the relative permeabilities

$$k_{rw} = k_{rw}(s_w), \qquad k_{ro} = k_{ro}(s_w, s_g), \qquad k_{rg} = k_{rg}(s_g).$$

Typical relative permeability curves are given in Figs. 3 and 4. Then equation (4.19) can be simplied further:

$$\lambda_w \frac{\partial \lambda}{\partial s_g}\,\frac{\partial p_{cwo}}{\partial s_w} = \lambda_g \frac{\partial \lambda}{\partial s_w}\,\frac{\partial p_{cgo}}{\partial s_g}. \tag{4.20}$$

We can construct three-phase relative permeability and capillary pressure curves which satisfy the condition (4.20). A simple numerical procedure for constructing these curves has been described in [8]. Some of the numerical examples satisfying (4.20) have been compared with the classical Stone's model [23], which does not satisfy this condition, and similar results were obtained. Here we will see that some simplified three-phase models in fact (or approximately) satisfy the condition (4.20).

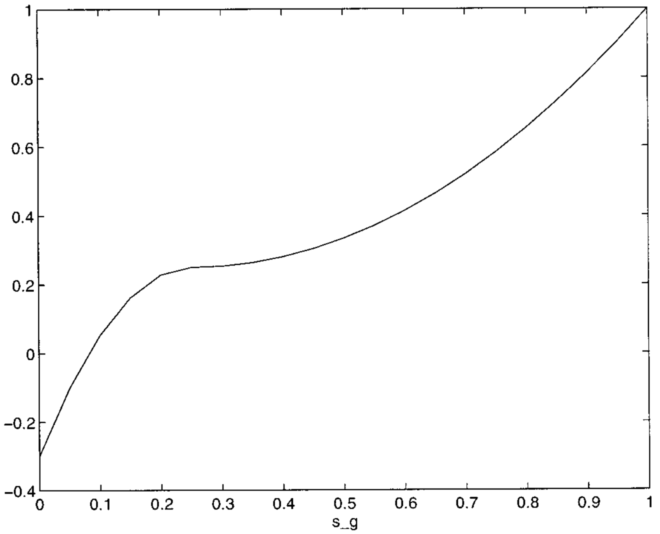

FIG. 2. The typical normalized capillary pressure p_{cgo}.

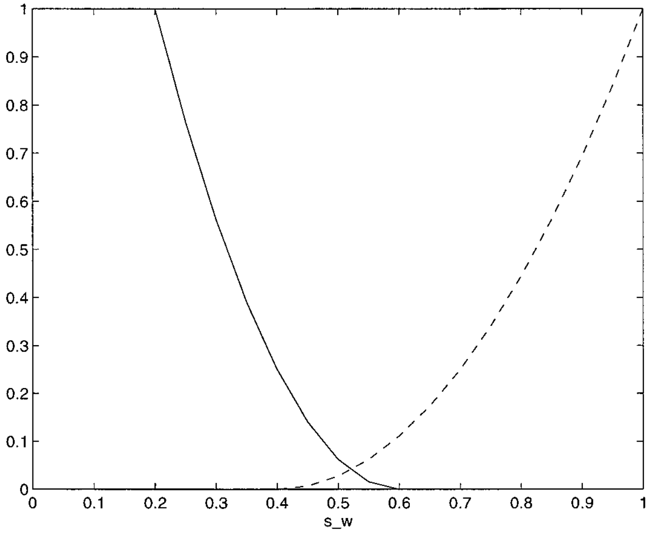

FIG. 3. Typical water-oil imbibition relative permeabilities k_{ro} (left) and k_{rw} (right).

It follows from (4.20) that, if the total mobility λ is close to a constant function, then the total differential condition holds approximately. We now consider the following cases. The simplified Corey and Baker models [15, 16] for the three-phase relative permeabilities have the form

$$k_{rw} = s_w^{e_w}, \qquad k_{ro} = (1 - s_w - s_g)^{e_o}, \qquad k_{rg} = s_g^{s_g},$$

where the e_α's are constants. Substitute them into (4.20) to see that

$$\begin{aligned}\frac{s_w^{e_w}}{\mu_w}&\left\{\frac{e_g}{\mu_g}s_g^{e_g-1} - \frac{e_o}{\mu_o}(1-s_w-s_g)^{e_o-1}\right\}\frac{\partial p_{cwo}}{\partial s_w}\\ &= \frac{s_g^{e_g}}{\mu_g}\left\{\frac{e_w}{\mu_w}s_w^{e_w-1} - \frac{e_o}{\mu_o}(1-s_w-s_g)^{e_o-1}\right\}\frac{\partial p_{cgo}}{\partial s_g}.\end{aligned} \tag{4.21}$$

Now, we see that the total differential condition is identically satisfied in the case of compressible fluids with cross-relative permeabilities where $e_\alpha = 1, \alpha = w, o, g$, and unity viscosities. For other choices of e_α, Eq. (4.21) depends on the definitions of the capillary pressures p_{cwo} and p_{cgo}, and can be approximately (if not identically) satisfied by appropriate choices of parameters.

5. PSEUDO-GLOBAL FORMULATION

The global formulation in Section 4 requires the total differential condition (4.3) on the shape of three-phase relative permeability and capillary pressure functions. In this section we derive a pseudo-global pressure-saturation formulation, which does not require this condition.

5.1. *Pseudo-global Formulation with the Total Velocity.* Again, the phase and total mobilities and the fractional flow functions are defined as in Subsection 3.1:

$$\lambda_\alpha = k_{r\alpha}/\mu_\alpha, \qquad \lambda = \sum_\beta \lambda_\beta, \quad f_\alpha = \lambda_\alpha/\lambda, \quad \alpha = w, o, g.$$

To introduce a pseudo-global pressure, we assume that the fractional flow functions f_α depend solely on the saturations s_w and s_g (for pressure-dependent functions f_α, see the next subsection). Also, assume that the capillary pressures satisfy (4.18). Then it follows from (3.3a) that

$$u = -k\lambda\left(\nabla p_o - G_\lambda + \sum_\alpha f_\alpha \frac{dp_{c\alpha o}}{ds_\alpha}\nabla s_\alpha\right). \tag{5.1}$$

We introduce the mean values

$$\bar{f}_w(s_w) = \frac{1}{1-s_w}\int_0^{1-s_w} f_w(s_w, \xi)\,d\xi,$$

$$\bar{f}_g(s_g) = \frac{1}{1-s_g}\int_0^{1-s_g} f_g(\xi, s_g)\,d\xi,$$

and the pseudo-global pressure

$$\begin{aligned}p = p_o &+ \int_{s_{wc}}^{s_w} \bar{f}_w(\xi)\frac{dp_{cwo}(\xi)}{ds_w}\,d\xi\\ &+ \int_{s_{gc}}^{s_g} \bar{f}_g(\xi)\frac{dp_{cgo}(\xi)}{ds_g}\,d\xi,\end{aligned}$$

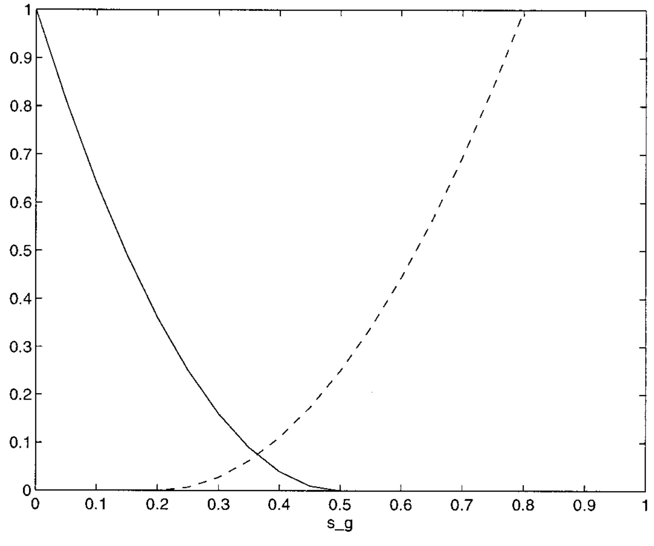

FIG. 4. Typical gas-oil drainage relative permeabilities k_{ro} (left) and k_{rg} (right).

where s_{wc} and s_{gc} are such that $p_{cwo}(s_{wc}) = 0$ and $p_{cgo}(s_{gc}) = 0$. Now, by (5.1), we see that

$$u = -k\lambda\Big(\nabla p - G_\lambda + \sum_\alpha (f_\alpha - \bar{f}_\alpha)\frac{dp_{c\alpha o}}{ds_\alpha}\nabla s_\alpha\Big), \qquad (5.2)$$

where $\bar{f}_o = 0$. Equations (3.3b) and (3.4) remain the same here.

5.2. *Pseudo-global Formulation with the Total Flux.* The phase and total mobilities and the fractional flow functions are again given as in Subsection 3.2:

$$\lambda_\alpha = k_{r\alpha}\rho_\alpha/\mu_\alpha, \qquad \lambda = \sum_\beta \lambda_\beta,\ f_\alpha = \lambda_\alpha/\lambda, \quad \alpha = w, o, g.$$

We now assume that the fractional flow functions f_α depend on the saturations s_w and s_g and a pressure p. Then the mean values are accordingly modified by

$$\bar{f}_w(s_w, p) = \frac{1}{1 - s_w}\int_0^{1-s_w} f_w(s_w, \xi, p)\,d\xi,$$

$$\bar{f}_g(s_g, p) = \frac{1}{1 - s_g}\int_0^{1-s_g} f_g(\xi, s_g, p)\,d\xi,$$

and the pseudo-global pressure has the corresponding expression

$$p = p_o + \int_{s_{wc}}^{s_w} \bar{f}_w(\xi, p)\frac{dp_{cwo}(\xi)}{ds_w}\,d\xi + \int_{s_{gc}}^{s_g} \bar{f}_g(\xi, p)\frac{dp_{cgo}(\xi)}{ds_g}\,d\xi.$$

Apply this definition to (3.7a) to obtain

$$u = -k\lambda\left(\omega\nabla p - G_\lambda + \sum_\alpha (f_\alpha - \bar{f}_\alpha)\frac{dp_{c\alpha o}}{ds_\alpha}\nabla s_\alpha\right), \qquad (5.3)$$

where

$$\omega = 1 - \int_{s_{wc}}^{s_w} \frac{d}{dp}\bar{f}_w(\xi, p)\frac{dp_{cwo}(\xi)}{ds_w}\,d\xi - \int_{s_{gc}}^{s_g} \frac{d}{dp}\bar{f}_g(\xi, p)\frac{dp_{cgo}(\xi)}{ds_g}\,d\xi.$$

The other two equations (3.7b) and (3.8) remain unchanged.

6. THEORETICAL COMPARISON

We first note that if f_w and f_g are close to their respective mean values $\bar{f}_w$ and $\bar{f}_g$, then the last term in the right-hand side of (5.2) and (5.3) can be neglected. In particular, in the case of compressible fluids with cross-relative permeabilities and unity viscosities mentioned in Subsection 4.3, $f_w = \bar{f}_w$ and $f_g = \bar{f}_g$. Hence, in these cases the pseudo-global form reduces to the global form. However, in the general case the pseudo-global form is essentially the same as the phase form, and the coupling between the pressure and saturation equations in these two forms has the same pattern. Here we compare the phase and global forms; an analogous comparison between the pseudo-global and global forms can be carried out. The comparison between the phase and pseudo-global forms will be given in the next section numerically.

We compare Eqs. (3.3) and (3.4) with Eqs. (4.7) and (4.8); the same comparison between Eqs. (3.7) and (4.15), and Eqs. (3.8) and (4.16) can be done in the same way. Note that the "continuity" equations (3.3b) and (4.7b) have the same form. However, the coupling between the pressure and saturation equations in (3.3) and (3.4) is stronger than that between the equations in (4.7) and (4.8). In particular, Eq. (3.3a) has the gradient of the two capillary pressure functions p_{cwo} and p_{cgo}, with different coefficients, but Eq. (4.7a) in form resembles the Darcy law for the single-phase flow, and is much simpler. Hence the computation of the pressure equation (3.3a) by the mixed finite element methods described in the next section requires the approximation of the two terms

$$f_w\nabla p_{cwo} \qquad \text{and} \qquad f_g\nabla p_{cgo}.$$

What is more, while the capillary diffusion terms involve the gradient of the two capillary pressure functions p_{cwo} and p_{cgo} in (3.4) for $\alpha = w$ and g, the calculation of the diffusion terms requires the resolution of the four linear systems

$$(\lambda_o + \lambda_g)\lambda_w\nabla p_{cwo}, \qquad \lambda_g\lambda_w\nabla p_{cgo}, \qquad \lambda_w\lambda_g\nabla p_{cwo},$$
$$\text{and} \qquad (\lambda_o + \lambda_w)\lambda_g\nabla p_{cgo}.$$

However, in (4.8) for $\alpha = w$ and g the diffusion terms only require the resolution of the two linear systems

$$\lambda_w\nabla(p_c - p_{cwo}) \qquad \text{and} \qquad \lambda_g\nabla(p_c - p_{cgo}).$$

Therefore, we would expect that Eqs. (3.3) and (3.4) are more expensive from the computational point of view. This is the case, as shown in the next section.

We close this section with two remarks. First, in the uninteresting case in which $p_{cwo} = p_{cgo} \equiv 0$, the models presented in Subsections 3.1, 4.1, and 5.1 (respectively, Subsections 3.2, 4.2, and 5.2) are the same. Second, boundary conditions imposed for the three-phase flow equations can be incorporated into the fractional flow formulation in the same manner as for the two-phase flow [14].

7. NUMERICAL COMPARISON

In this section we compare the previous three formulations for two sets of data. The first set of data is relatively simple and satisfies the total differential condition (4.3). Thus in this case we numerically compare the phase and global formulations. The second set of data is more physically adequate, but does not satisfy the condition (4.3). For this set of data we compare the phase and pseudo-global formulations.

7.1. *The First Test.* The capillary pressure functions are defined as

$$p_{cwo} = s_w - 1, \qquad p_{cgo} = 1 - s_g.$$

Recall that p_{cwo} is the negative water phase capillary pressure. The relative permeability curves are given by

$$k_{rw} = s_w, \qquad k_{ro} = 1 - s_w - s_g, \qquad k_{rg} = s_g.$$

With these choices, the total differential condition (4.3) is satisfied from the discussion in Subsection 4.3. Further, with $\phi = \mu_\alpha = \rho_\alpha = 1$, $\alpha = w, o, g$, and $\tilde{g} = 0$, the mobility and fractional flow functions become

$$\lambda_w = f_w = s_w, \qquad \lambda_o = f_o = 1 - s_w - s_g,$$
$$\lambda_g = f_g = s_g, \qquad \lambda = 1.$$

Thus the function p_c is given by

$$p_c = -\tfrac{1}{2}(1 - s_w^2 + s_g^2).$$

For the present set of data, the system in (3.3) and (3.4) (respectively, (4.7) and (4.8)) is the same as that in (3.7) and (3.8) (respectively, (4.15) and (4.16)). Finally, the domain Ω is the unit cube $\Omega = (0, 1)^3$, and a no-flow boundary condition for each phase is taken,

$$u_\alpha \cdot \nu = 0, \qquad \alpha = w, o, g, x \in \partial\Omega, t > 0, \tag{7.1}$$

where ν is the outer unit normal to the boundary $\partial\Omega$ of Ω.

TABLE I

Convergence of p_h for the Phase System in Test One

$1/h$	L^∞-error	L^∞-order
10	0.10356	—
20	0.05147	1.01
40	0.02533	1.02

In the phase pressure-saturation form, Eqs. (3.3) and (3.4) now reduce to (with $x \in \Omega$ and $t > 0$)

$$u = -k(\nabla p + s_w \nabla s_w - s_g \nabla s_g), \tag{7.2a}$$
$$\nabla \cdot u = q, \tag{7.2b}$$

and

$$\frac{\partial s_w}{\partial t} + \nabla \cdot \{s_w u - k s_w ((1 - s_w)\nabla s_w + s_g \nabla s_g)\} = q_w, \tag{7.3a}$$
$$\frac{\partial s_g}{\partial t} + \nabla \cdot \{s_g u + k s_g ((1 - s_g)\nabla s_g + s_w \nabla s_w)\} = q_g, \tag{7.3b}$$

where $q = \Sigma_\alpha q_\alpha$. Similarly, in the global pressure-saturation form the pressure equation (4.7) becomes

$$u = -k\nabla p, \tag{7.4a}$$
$$\nabla \cdot u = q; \tag{7.4b}$$

the saturation equations are the same as in (7.3) for the present set of data. Recall that the p in (7.2a) is the oil phase pressure, while the p in (7.4a) is the global pressure defined in (4.5). Also, it follows from (7.1) that the boundary condition for (7.2) and (7.4) is

$$u \cdot \nu = 0, \qquad x \in \partial\Omega, t > 0, \tag{7.5}$$

and that the boundary conditions for (7.3a) and (7.3b) are

$$\{k s_w ((1 - s_w)\nabla s_w + s_g \nabla s_g)\} \cdot \nu = 0, \qquad x \in \partial\Omega, t > 0, \tag{7.6a}$$
$$\{k s_g ((1 - s_g)\nabla s_g + s_w \nabla s_w)\} \cdot \nu = 0, \qquad x \in \partial\Omega, t > 0. \tag{7.6b}$$

TABLE II

Convergence of s_h for the Phase System in Test One

$1/h$	L^∞-error	L^∞-order
10	0.23302	—
20	0.11875	0.97
40	0.05950	0.99

TABLE III

Convergence of p_h for the Global System in Test One

$1/h$	L^∞-error	L^∞-order
10	0.10402	—
20	0.05208	1.00
40	0.02576	1.02

TABLE V

CPU Times up to $t = 1$ in Test One

$1/h$	CPU-B	CPU-P
10	0.51	0.93
20	4.02	7.39
40	16.08	29.53

For the present simple problem, (7.4) implies that the pressure equation is completely decoupled from the saturation equations in the global pressure-saturation form, and thus it can be independently computed and the resulting total velocity can be used by the saturation equations later. In the phase pressure-saturation form, the system in Eqs. (7.2) and (7.3) is solved sequentially. An approximation of u is first obtained at time level $t = t^n$ from solution of Eq. (7.2) with the saturations s_w and s_g evaluated at the previous time level $t = t^{n-1}$. Then, using the current approximation for u, approximations of s_w and s_g are obtained at $t = t^n$ by using (7.3a) and (7.3b) simultaneously. The saturation equations are solved here by the classical upweighting finite difference scheme, while the pressure equation is solved by a mixed finite element method. We assume that the reader is familiar with the former scheme; the latter method will be reviewed in the Appendix.

Uniform partitions of Ω into rectangular parallelepipeds with the space step $h = \Delta x = \Delta y = \Delta z$ are taken. The time differentiation terms in (7.3) are discretized with the backward Euler scheme, and the time step is assumed to be proportional to the space step: $\Delta t = \kappa h$, where κ is the proportionality constant. A cell-centered finite difference method with the seven point stencil is used for the solution of the saturation equations, while a mixed finite element method with the use of the Raviart–Thomas–Nedelec mixed space [22, 20] of lowest-order over rectangular parallelepipeds is applied to the solution of the pressure equations (see the Appendix). Tables I–IV describe the errors and convergence orders in the L^∞-norm for the pressure and saturation at $t = 1$ for the phase and global pressure-saturation differential systems, where s_h is the approximation to the water saturation. In Table V, the CPU times in minutes for solving the whole pressure-saturation system over the given mesh up to time $t = 1$ from the initial time $t = 0$ are presented. CPU-P denotes the CPU times for the phase system, while CPU-B indicates those for the global system. All experiments are carried out on a Sun workstation.

It follows from Tables I–IV that the numerical results agree with the theoretical error prediction $O(\Delta t + h)$ for both systems. However, the CPU times required for the solution of the phase pressure-saturation system almost double those for the global system. This shows that the latter system can be more easily solved, and agrees with our theoretical observation in Section 6. Lots of time is spent on the coupling between the pressure and saturation equations in the former system.

7.2. *The Second Test.* In the second example we test a more physically adequate set of data for the comparison between the phase and pseudo-global formulations. The relative permeability curves are given by the modified Corey's model [15]

$$\begin{aligned} k_{rw} &= 0.21 s_w^{1.5}, \\ k_{ro} &= 0.71 s_o^{2.5}(1 - (1 - s_o)^{2.5}), \\ k_{rg} &= s_g^{2.5}(1 - (1 - s_g)^{2.5}), \end{aligned} \tag{7.7}$$

where the reduced saturations s_α are related to the phase saturations $\tilde{s}_\alpha$ by (2.4) with the residual saturations $\tilde{s}_{r\alpha}$ given by

$$\tilde{s}_{rw} = 0.25,$$

$$\tilde{s}_{ro} = 0.35,$$

$$\tilde{s}_{rg} = 0.05.$$

TABLE IV

Convergence of s_h for the Global System in Test One

$1/h$	L^∞-error	L^∞-order
10	0.19847	—
20	0.09982	0.99
40	0.04977	1.00

TABLE VI

Convergence of p_h for the Phase System in Test Two

$1/h$	L^∞-error	L^∞-order
10	0.31076	—
20	0.17043	0.87
40	0.08821	0.95

TABLE VII

Convergence of s_h for the Phase System in Test Two

$1/h$	L^∞-error	L^∞-order
10	0.49217	—
20	0.25570	0.94
40	0.13052	0.97

TABLE VIII

Convergence of p_h for the Pseudo-global System in Test Two

$1/h$	L^∞-error	L^∞-order
10	0.14921	—
20	0.07840	0.93
40	0.03923	1.00

The capillary pressure functions are determined by

$$\begin{aligned} p_{cwo} &= -a_w(1-s_w)^{b_w}, \\ p_{cgo} &= (1-s_g)^{b_g}(a_g(s_g^{-1}-1)^{c_g}+\pi_g), \end{aligned} \tag{7.8}$$

where the constants a_w, etc., depend on the residual saturations. The water and oil densities are taken to be 1000 kg/m^3 and 1200 kg/m^3, respectively, while the gas density is chosen as

$$\rho_g = \rho_{0g}\left(1+\frac{p_g}{p_{0g}}\right), \tag{7.9}$$

where ρ_{0g} is the density of the gas phase at the reference pressure p_{0g}. The constants in (7.8) and (7.9) are not important for the present test, and are chosen randomly. Finally, the viscosities of the water, oil, and gas phases are 1 cp, 0.9 cp, and 0.8 cp, respectively, and ϕ and $\tilde{g}$ are the same as in the first example. The no-flow boundary condition for each phase in (7.1) is also exploited here.

The comparison is here done between the system given by (3.3) and (3.4) and that given by (5.2), (3.3b), and (3.4); similar results have been observed for the system in (3.7) and (3.8) and that in (5.3), (3.7b), and (3.8). Note that, in the present situation, the pressure equations are parabolic, and are not decoupled from their saturation equations. These are the differences between the two test cases. Another difference is, as mentioned above, that the functions in (7.7) and (7.8) do not satisfy the condition (4.3).

The same discretization techniques and set of numerical data in the first example are used here. The integrals in the pseudo-global system are computed numerically by a scheme which is consistent with the used discretization schemes in terms of convergence order. The convergence results and CPU times are displayed in Tables VI–X for the present comparison. The convergence results have the same performance as in the first example. However, it turns out that the pseudo-global system takes more time, which is indicated in Table X by CPU-S. The reason for this is that the coupling between the pressure and saturation equations in the phase and pseudo-global forms has the same pattern, but extra time is needed to handle the numerical integrals in the latter form. As mentioned before, in the case where f_w and f_g are close to their respective mean values $\overline{f}_w$ and $\overline{f}_g$, the pseudo-global form is very useful. For, in this case, this form approximates the global form, and thus the pressure equation is more decoupled from the saturation equations. We have observed this in our numerical experiments (not reported here).

We end with two remarks. First, the transport term in the saturation equations is governed by the total velocity (respectively, flux) u. Thus accurate numerical simulations require an accurate approximation for u. The mixed finite element method is here used to approximate u and p simultaneously, and produces an accurate velocity [18]. Second, due to their convection-dominated feature, more efficient approximate procedures should be used to solve the saturation equations. However, the interest here is in the comparison between the two differential systems; the simple finite difference scheme is accurate for this purpose [12].

8. CONCLUDING REMARK

The phase, pseudo-global, and global pressure-saturation differential systems have been established for the three-phase fluid flow in porous media. Comparisons between these systems have been carried out both theoreti-

TABLE IX

Convergence of s_h for the Pseudo-global System in Test Two

$1/h$	L^∞-error	L^∞-order
10	0.28397	—
20	0.15026	0.92
40	0.07911	0.93

TABLE X

CPU Times up to $t = 1$ in Test Two

$1/h$	CPU-S	CPU-P
10	2.33	1.87
20	18.46	14.82
40	73.78	59.26

cally and numerically. The global differential system is far more efficient than the phase and pseudo-global systems from the computational point of view, and also more suitable for mathematical analysis. The advantage of the global form can be more obviously seen in the case of incompressible flow and one-space dimension. For, in this case, the global pressure equation can be analytically solved. The weakness of the global formulation is the need of the satisfaction of the total differential condition by the three-phase relative permeability and capillary pressure curves. In general, the phase formulation is useful. However, if the fractional flow functions of the water and gas phases are close to their respective mean values, the pseudo-global system formulation is more useful.

APPENDIX: REMARKS ON A MIXED METHOD

We rewrite Eqs. (7.4) and (7.5) as

$$u = -k\nabla p, \qquad x \in \Omega, \tag{A.1a}$$

$$\nabla \cdot u = q, \qquad x \in \Omega, \tag{A.1b}$$

$$u \cdot \nu = 0, \qquad x \in \partial\Omega. \tag{A.1c}$$

For compatibility, $q(x)$ needs to satisfy the condition

$$\int_\Omega q(x)dx = 0.$$

Define the spaces

$$\begin{aligned} L^2(\Omega) &= \left\{ w : \int_\Omega |w(x)|^2 dx < \infty \right\}, \\ H(\mathrm{div};\Omega) &= \{ v \in (L^2(\Omega))^3 : \nabla \cdot v \in L^2(\Omega)\}, \\ W &= \{ w \in L^2(\Omega) : \int_\Omega w(x)dx = 0\}, \\ V &= \{ v \in H(\mathrm{div};\Omega) : v \cdot \nu = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Then the mixed form of (A.1) for the pair $(u, p) \in V \times W$ is

$$(\nabla \cdot u, w) = (q, w), \qquad \forall w \in W, \tag{A.2a}$$

$$(k^{-1}u, v) - (p, \nabla \cdot v) = 0, \qquad \forall v \in V, \tag{A.2b}$$

where $(\cdot, \cdot)$ is the $L^2(\Omega)$ or $(L^2(\Omega))^3$ inner product, as appropriate. This system has a unique solution [7].

For $0 < h < 1$, let $\mathscr{E}_h$ be a partition of Ω into rectangular parallelepipeds. In $\mathscr{E}_h$ we need that adjacent elements completely share their common face. Then we introduce the Raviart–Thomas–Nedelec mixed space [22, 20] of lowest-order

$$\begin{aligned} V_h = \{ v \in V : v|_E &= (a_E^1 + a_E^2 x, a_E^3 + a_E^4 y, a_E^5 + a_E^6 z), a_E^i \in \Re, \forall E \in \mathscr{E}_h\}, \\ W_h = \{ w \in W : v|_E &= b_E, b_E \in \Re, \forall E \in \mathscr{E}_h\}. \end{aligned}$$

Then the mixed finite element solution of (A.2) is $(u_h, p_h) \in V_h \times W_h$ satisfying

$$(\nabla \cdot u_h, w) = (q, w), \qquad \forall w \in W_h, \tag{A.3a}$$

$$(k^{-1}u_h, v) - (p_h, \nabla \cdot v) = 0, \qquad \forall v \in V_h. \tag{A.3b}$$

Again, this system has a unique solution [7].

The linear system arising from (A.3) is a saddle point problem [7], which can be expensive to solve. One of useful numerical methods for solving this saddle point problem is the inexact Uzawa algorithm (see, e.g., [6, 17]). A more efficient approach was suggested by means of a nonmixed formulation. Namely, it has been shown that the mixed finite element method is equivalent to a modification of a nonconforming Galerkin method [2, 3, 9, 10]. The nonconforming method yields a symmetric and positive definite problem, which can be more easily solved.

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